Math 6261 23-03-10

Review.

(Lévy's Convergence Thm):

Let (F_n) be a sequence of DFs with CFs φ_n. Suppose that

$$\lim_{n\to\infty} \varphi_n(\theta) = g(\theta)$$

for some function $g: \mathbb{R} \to \mathbb{C}$. Furthermore, suppose g is cts at $\theta = 0$.

Then $F_n \xrightarrow{w} F$ for some CF F on $\mathbb{R}$ and $g = \varphi_F$.

§ 3.3. The central limit theorem.

Thm 3.11 (CLT) Let (X_n) be an i.i.d. sequence of r.v.'s with

$$E(X_n) = 0 \quad \text{and} \quad Var(X_n) = \sigma^2.$$

Write $S_n = X_1 + \cdots + X_n$. Then for any $x \in \mathbb{R}$,

$$P\left(\frac{S_n}{\sqrt{n}\,\sigma} \leq x \right) \longrightarrow \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

as $n \to \infty$.

(standard normal distribution).

To prove the CLT, we give some Lemmas.

Lem 3.12. The CF φ of the standard normal distribution is given by

$$\varphi(\theta) = e^{-\frac{1}{2}\theta^2}.$$

Pf. Notice that by def,

$$\varphi(\theta) = \int_{\mathbb{R}} e^{i\theta x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$

$$\varphi'(\theta) = \int_{\mathbb{R}} ix\, e^{i\theta x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$= \int_{\mathbb{R}} -i e^{i\theta x} \cdot \frac{1}{\sqrt{2\pi}} \, d\left(e^{-x^2/2}\right)$$

$$= \int_{\mathbb{R}} \frac{i}{\sqrt{2\pi}} e^{-(x^2/2)} \, d\, e^{i\theta x} \qquad (\text{integration by part})$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} -\theta\, e^{i\theta x} e^{-x^2/2} \, dx$$

$$= -\theta\, \varphi(\theta)$$

Hence $\left(\ln \varphi(\theta)\right)' = -\theta \quad \Rightarrow \quad \varphi(\theta) = c\, e^{-\theta^2/2}.$

But $\varphi(0) = 1$. So $\varphi(\theta) = e^{-\theta^2/2}$.

$\blacksquare$

Lem 3.13. For $n=0,1,\cdots,$ and $x\in\mathbb{R}$, write

$$R_n(x) = e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}$$

Then

$$|R_n(x)| \le \min\left\{ \frac{2|x|^n}{n!}, \frac{|x|^{n+1}}{(n+1)!} \right\}.$$

Pf. Notice that

$$R_0(x) = e^{ix} - 1 = \int_0^x i e^{iy}\, dy.$$

From these two expressions we obtain

$$|R_0(x)| \le \min\{2, |x|\}.$$

Since

$$R_n(x) = \int_0^x i\, R_{n-1}(y)\, dy \qquad \text{for } n\ge 1,$$

We obtain the desired inequality by induction. $\square$

Lem 3.14. For $|z| < \frac{1}{2}$,

$$|\log(1+z) - z| < |z|^2.$$

pf. $$\log(1+z) - z = \int_0^z \frac{-w}{1+w}\, dw$$

$$= z^2 \int_0^1 \frac{-t\, dt}{1+zt}$$

Hence $\left| \log(1+z) - z \right| \leq |z|^2 \int_0^1 2t\, dt = |z|^2$.

Pf of the CLT:

Let φ denote the CF of X. Recall that $EX=0$, $EX^2=\sigma^2$.

Then $\varphi(\theta) - \left(1 - \frac{1}{2}\sigma^2\theta^2\right)$

$$= \int e^{i\theta x} d\mu(x) - \int \left(1 + i\theta x + \frac{(i\theta x)^2}{2}\right) d\mu(x)$$

$$= \int R_2(\theta x)\, d\mu(x)$$

By Lem 3.13,

$$|R_2(\theta x)| \leq \min\left\{ \frac{2\,|\theta x|^2}{2!}, \frac{|\theta x|^3}{3!} \right\}$$

$$= \theta^2 \cdot \min\left\{ |x|^2, \frac{\theta\,|x|^3}{6} \right\}.$$

It follows that

$$\left| \varphi(\theta) - \left(1 - \tfrac{1}{2}\sigma^2\theta^2\right) \right| \leq \theta^2 \cdot \int \min\left\{ |x|^2, \frac{\theta\,|x|^3}{6} \right\} d\mu(x)$$

By the DCT, $\lim_{\theta\to 0} \int \min\left\{ |x|^2, \frac{\theta |x|^3}{6} \right\} d\mu(x) = 0$.

It follows that

$$\varphi(\theta) = 1 - \frac{1}{2}\sigma^2\theta^2 + o(\theta^2).$$

Note that

$$\varphi_{\frac{S_n}{\sigma\sqrt{n}}}(\theta) = \varphi\left(\frac{\theta}{\sigma\sqrt{n}}\right)^n$$

$$= \left(1 - \frac{1}{2}\frac{\theta^2}{n} + o\left(\frac{\theta^2}{n}\right)\right)^n$$

Hence

$$\log \varphi_{\frac{S_n}{\sigma\sqrt{n}}}(\theta) = n \log\left(1 - \frac{1}{2}\frac{\theta^2}{n} + o\left(\frac{\theta^2}{n}\right)\right)$$

By Lem 3.14,

$$\log\left(1 - \frac{1}{2}\frac{\theta^2}{n} + o\left(\frac{\theta^2}{n}\right)\right) - \frac{1}{2}\frac{\theta^2}{n} - o\left(\frac{\theta^2}{n}\right)$$

$$= o\left(\frac{\theta^4}{n^2}\right)$$

Hence

$$n \log\left(1 - \frac{1}{2}\frac{\theta^2}{n} + o\left(\frac{\theta^2}{n}\right)\right) \longrightarrow -\frac{1}{2}\theta^2.$$

That is,

$$\varphi_{\frac{S_n}{\sigma\sqrt{n}}}(\theta) \longrightarrow e^{-\frac{1}{2}\theta^2}.$$

By Lévy's convergence thm,

$$P\left(\frac{S_n}{\sigma\sqrt{n}} \leqslant x\right) \longrightarrow \Phi(x). \qquad \blacksquare$$

Remark: Pairwise independence is good enough for the SLLN. However it is not good enough for the CLT; see the example below.

Example Let $\xi_1, \cdots, \xi_n, \cdots,$ be i.i.d with

$$P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}.$$

Set $S_{2^n} = \xi_1 (1+\xi_2)(1+\xi_3)\cdots(1+\xi_{n+1})$

$$= \begin{cases} 2^n & \text{with prob. } 2^{-n-1}, \\ 0 & \text{with prob. } 1-2^{-n-1}. \end{cases}$$

Notice that S_{2^n} is the sum of 2^n r.v.'s

$\xi_1, \ \xi_1\xi_2, \ \xi_1\xi_3, \ \cdots$ (the terms in the expansion of the product in defining S_{2^n})

which are pairwise independent, each of them has the same distribution as ξ_i.

Clearly $\dfrac{S_{2^n}}{\sqrt{2^n}}$ does not converge weakly to the standard normal distribution.

- The following result gives the rate of convergence in the CLT

Thm 3.15. (Berry - Esseen Thm).

Let $X_1, X_2, \cdots,$ be i.i.d. with $EX_i = 0$, $EX_i^2 = \sigma^2$ and

$$E|X_i|^3 = \rho < \infty.$$

Then

$$\left| P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{3\rho}{\sigma^3 \cdot \sqrt{n}} \quad \text{for all } x \in \mathbb{R}.$$

- Below is a famous result of Kolmogrov on the convergence rate in the SLLN.

Thm 3.16 (Kolmogrov's law of iterated logarithm)

Let $X_1, X_2, \cdots,$ be i.i.d. with $E(X_i) = 0$ and $Var(X_i) = 1$.

Then almost surely,

$$\overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n \log\log n}} = 1, \qquad \underline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{2n \log\log n}} = -1,$$

where $S_n = X_1 + \cdots + X_n$.

Below we state the CLT in $\mathbb{R}^d$.

Thm 3.17. Let $X_1, X_2, \cdots,$ be i.i.d. random vectors in $\mathbb{R}^d$ with $EX_n = \mu$, and finite covariances

$$\Gamma_{i,j} = E\Big((X_{n,i} - \mu_i)(X_{n,j} - \mu_j)\Big) \qquad \text{for } 1 \le i, j \le d.$$

Set $S_n = X_1 + \cdots + X_n$. Then

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{w} \mathcal{X},$$

where $\mathcal{X}$ has a multivariate normal distribution with mean 0 and covariance $\Gamma = (\Gamma_{i,j})$.